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Summary. We propose to regularize the convective term in the Navier-Stokes equations in such a manner that the symmetries that correspond to the invariance of the energy, the enstrophy (in 2D) and helicity are preserved. The underlying idea is to restrain the convective production of smaller and smaller scales of motion by means of vortex stretching in an unconditional stable manner, meaning that the solution can not blow up (in the energy-norm). The numerical algorithm used to solve the governing equations preserves the symmetry properties too. The simulation shortcut is successfully tested for a turbulent channel flow ($\mathrm{Re}_\tau = 180$).

1 Introduction

Most turbulent flows can not be computed directly from the (incompressible) Navier-Stokes equations,

$$\partial_t u + \mathcal{C}(u, u) + \mathcal{D}(u) + \nabla p = 0, \tag{1}$$

because they possess far too many scales of motion. The computationally almost numberless small scales result from the nonlinear convective term $\mathcal{C}(u, v) = (u \cdot \nabla)v$ which allows for the transfer of energy from scales as large as the flow domain to the smallest scales that can survive viscous dissipation. As the full energy cascade can not be computed, a dynamically less complex mathematical formulation is sought. In the quest for such a formulation, we consider smooth approximations (regularizations) of the convective term:

$$\partial_t u_\epsilon + \widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon) + \mathcal{D}(u_\epsilon) + \nabla p_\epsilon = 0, \tag{2}$$

where the variable name is changed from u to u_ϵ to stress that the solution of (2) differs from that of (1). Notice that by filtering (2) and comparing the result term-by-term with the equation governing the dynamics of the filtered velocity $\overline{u}_\epsilon$ in LES, we may identify the closure model, see also [1]:

$$\text{closure model}(\overline{u}_\epsilon) = \mathcal{C}(\overline{u}_\epsilon, \overline{u}_\epsilon) - \overline{\widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon)}.$$

Here, we want to smooth the convective term directly to set bounds to the creation of smaller and smaller scales of motion and thus to confine the cascade of energy. That is, the low modes of the solution u_ϵ of (2) should approximate the corresponding low modes of the solution u of the Navier-Stokes equations (1), whereas the high modes of u_ϵ should vanish much faster than those of u. In that case, Eq. (2) provides a basis for a simulation shortcut.

The first outstanding approach in this direction goes back to Leray [2], who took $\widetilde{\mathcal{C}}(u,u) = \mathcal{C}(\overline{u},u)$ and proved that a moderate filtering of the convective velocity is sufficient to regularize a turbulent flow. Cheskidov et al. [3] have analyzed Leray's approximation for a Helmholtz filter. They show that the complexity of the 3D Leray model lies between that of the 2D and 3D Navier-Stokes equations. The Navier-Stokes-alpha-model forms another example of regularization modeling, see for instance [4],[5]. In this model, the convective term becomes $\widetilde{\mathcal{C}}_r(u,u) = \mathcal{C}_r(u,\overline{u})$, where $\mathcal{C}_r$ denotes the convective operator in rotational form: $\mathcal{C}_r(u,v) = (\nabla \times u) \times v$.

The regularization method basically alters the nonlinearity to control the convective energetic exchanges. In doing so, one can preserve certain fundamental properties of (the convective operator in) the Navier-Stokes equations exactly, e.g., symmetries, conservation properties, transformation properties, Kelvin's circulation theorem, Bernouilli's theorem, Karman-Howarth theorem, etc. [6].

In this paper, we propose to smooth $\mathcal{C}$ in such a manner that the symmetry properties that yield the invariance of the energy, the enstrophy (in 2D) and helicity are preserved. The underlying idea is to restrain the convective production of smaller and smaller scales of motion by means of vortex stretching, while ensuring that the solution does not blow up (in the energy-norm; in 2D also: enstrophy-norm). We anticipate that the unconditional stability enhances the accuracy at coarse resolutions. Here, it may be pointed out that the unconditional stability of $\widetilde{\mathcal{C}}$ allows for simulations at arbitrary coarse grids, provided the discretization of $\widetilde{\mathcal{C}}$ preserves the symmetry too, see for instance [7].

2 Symmetry-preserving smoothers

Approximations of particular interest are the ones that conserve the energy, the enstrophy (in 2D) and the helicity in the absence of viscous dissipation, among others because they are intrinsically stable (in the energy-norm; in 2D: enstrophy-norm). Note: the Leray model conserves the energy, but not the enstrophy or helicity, whereas the Navier-Stokes-alpha model conserves the enstrophy and helicity, yet not the energy.

The evolution of the energy follows from differentiating (u,u) with respect to time and rewriting $\partial_t u$ with the help of (1). In this way, we get a convective contribution given by the trilinear form $(\mathcal{C}(u,u),u)$. Since this form is skew-symmetric with respect to the last two arguments, that is

$$(\mathcal{C}(u,v),w) = -(v,\mathcal{C}(u,w)) \tag{3}$$

(see e.g. [8]), we have $(\mathcal{C}(u,v),v) = 0$ for any pair u, v, which implies that the convective contribution $(\mathcal{C}(u,u),u)$ cancels from the energy equation; hence the energy is conserved in the absence of viscous dissipation ($\mathcal{D} = 0$). Similarly it may be shown that (3) yields helicity-conservation. The evolution of the enstrophy is obtained by taking the inner product of the Navier-Stokes equations with the vector field $-\Delta u$. The resulting convective contribution vanishes in two spatial dimensions [8]: $(\mathcal{C}(u,u),\Delta u) = 0$ Actually, an even stronger form of enstrophy invariance holds [9]:

$$(\mathcal{C}(u,v),\Delta v) = (u,\mathcal{C}(\Delta v,v)). \tag{4}$$

Since the conservation of energy, enstrophy (in 2D) and helicity are intimately tied up with the symmetry properties (3) and (4) of the convective operator $\mathcal{C}$, we propose to approximate $\mathcal{C}$ in such manner that (3) and (4) are preserved exactly. This criterion yields the following class of approximations

$$\partial_t u_\epsilon + \mathcal{C}_n(u_\epsilon,u_\epsilon) + \mathcal{D}(u_\epsilon) + \nabla p_\epsilon = 0, \tag{5}$$

($n = 2, 4, 6$) in which the convective term is smoothened according to

$$\mathcal{C}_2(u,v) = \overline{\mathcal{C}(\overline{u},\overline{v})} \tag{6}$$
$$\mathcal{C}_4(u,v) = \mathcal{C}(\overline{u},\overline{v}) + \overline{\mathcal{C}(\overline{u},v')} + \overline{\mathcal{C}(u',\overline{v})} \tag{7}$$
$$\mathcal{C}_6(u,v) = \mathcal{C}(\overline{u},\overline{v}) + \mathcal{C}(\overline{u},v') + \mathcal{C}(u',\overline{v}) + \overline{\mathcal{C}(u',v')} \tag{8}$$

Here a bar denotes a filtered quantity and a prime indicates the residual. The three approximations $\mathcal{C}_n(u,u)$ are consistent with $\mathcal{C}(u,u)$, where the error is of the order of ϵ^n with $n = 2, 4, 6$ for symmetric filters with a filter length ϵ. Both the Leray model and the alpha model are second-order accurate in terms of ϵ.

The approximations (6)-(8) inherit the skew-symmetry of $\mathcal{C}$ by construction. That is, for any filter satisfying $(\overline{u},v) = (u,\overline{v})$, we have $(\mathcal{C}_n(u,v),w) = -(v,\mathcal{C}_n(u,w))$. They inherit the enstrophy invariance in 2D, $(\mathcal{C}_n(u,v),\Delta v) = (u,\mathcal{C}_n(\Delta v,v))$, provided the filter commutes with the Laplacian. Consequently, Eq. (5) conserves the energy, the enstrophy (in 2D) and the helicity if the filter satisfies both $(\overline{u},v) = (u,\overline{v})$ and $\overline{\Delta u} = \Delta\overline{u}$.

3 Vortex stretching

The evolution of the vorticity $\omega_\epsilon = \nabla \times u_\epsilon$ of any solution u_ϵ of Eq. (5),

$$\partial_t \omega_\epsilon + \mathcal{C}_n(u_\epsilon,\omega_\epsilon) + \mathcal{D}(\omega_\epsilon) = \mathcal{C}_n(\omega_\epsilon,u_\epsilon), \tag{9}$$

resembles that of the Navier-Stokes equations: the only difference is that $\mathcal{C}$ is replaced by the approximation $\mathcal{C}_n$. The Navier-Stokes equations give the source term

$$\mathcal{C}(\omega, u) = S\omega = \overline{S}\overline{\omega} + \overline{S}\omega' + S'\overline{\omega} + S'\omega' \tag{10}$$

where $S = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$ is the deformation tensor. The vortex stretching term $\mathcal{C}_n(\omega_\epsilon, u_\epsilon)$ in Eq. (9) reads

$$\mathcal{C}_2(\omega, u) = \overline{\overline{S}\overline{\omega}} \tag{11}$$

$$\mathcal{C}_4(\omega, u) = \overline{S}\overline{\omega} + \overline{\overline{S}\omega'} + \overline{S'\overline{\omega}} \tag{12}$$

$$\mathcal{C}_6(\omega, u) = \overline{S}\overline{\omega} + \overline{S}\omega' + S'\overline{\omega} + \overline{S'\omega'} \tag{13}$$

Qualitatively, vortex stretching leads to the production of smaller and smaller scales; hence to a continuous, local increase of both S' and ω'. Consequently, at the positions where vortex stretching occurs, the terms with S' and ω' will eventually amount considerably to the right-hand side of (10). Since these terms are diminished in (11)-(13), the conservative smoothing of the convective term counteracts the production of smaller and smaller scales by means of vortex stretching and may eventually stop the continuation of the vortex stretching process. So, in conclusion, the approximations $\mathcal{C}_n(u, u)$ restrain the convective production of smaller and smaller scales of motion by means of vortex stretching, while ensuring that the solution can not blow up (in the energy-norm; 2D: enstrophy-norm).

4 Triadic interactions

To study the interscale interactions in more detail, we continue in the spectral space, where we will restrict ourselves to the approximation $\mathcal{C}_4$; a similar analysis may be performed for $\mathcal{C}_2$ or $\mathcal{C}_6$. Taking the Fourier transform of (5)+(7) yields the evolution of each Fourier-mode $\widehat{u}_k(t)$ of u_ϵ:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{|k|^2}{\mathrm{Re}}\right)\widehat{u}_k + \mathcal{C}_k(\widehat{G}\widehat{u}, \widehat{G}\widehat{u}) + \widehat{G}_k\mathcal{C}_k(\widehat{G}\widehat{u}, (I-\widehat{G})\widehat{u}) + \widehat{G}_k\mathcal{C}_k((I-\widehat{G})\widehat{u}, \widehat{G}\widehat{u}) = 0,$$

where $\mathcal{C}_k(\widehat{u}, \widehat{u})$ denotes the spectral representation of the convective term in the Navier-Stokes equations and $\widehat{G}$ represents the Fourier transform of our filter. The Navier-Stokes dynamics is obtained by taking $\widehat{G} = I$.

The mode $\widehat{u}_k(t)$ interacts only with those modes whose wavevectors p and q form a trangle with the vector k. The local interactions between large scales (meaning that $\epsilon|k| < 1$ and $|k| \sim |p| \sim |q|$) approximate the Navier-Stokes dynamics up to $\mathcal{O}\left(\epsilon^4\right)$, i.e., the interactions between large scales are only slightly altered by the approximation $\mathcal{C}_4$. In order to investigate interactions involving longer wave-vectors (smaller scales of motion), the filter need be specified further. Since a Helmholtz filter enables a plain analysis of the interactions, we consider $\widehat{G}_k = (1 + \alpha^2|k|^2)^{-1}$ with $\alpha^2 = \epsilon^2/24$. For this filter, the spectral representation of $\mathcal{C}_4$ becomes

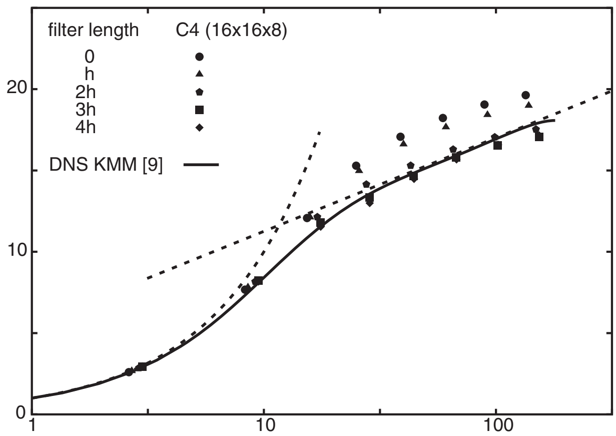
$$i\Pi(k) \sum_{p+q=k} \widehat{u}_p q \widehat{v}_q \frac{1 + \alpha^2(|k|^2 + |p|^2 + |q|^2)}{(1 + \alpha^2|k|^2)(1 + \alpha^2|p|^2)(1 + \alpha^2|q|^2)}$$

where $\Pi(k) = I - kk^{\mathrm{T}}/|k|^2$ denotes the projector onto divergence free velocity fields in the spectral space. By comparing this expression with the Navier-Stokes interactions, we see that all triad interactions are reduced by the application of the Helmholtz filter. The amount by which the triadic interactions are lessened depends on the length of the legs of the triangle $k = p+q$. The reduction is the largest for triangles with three long legs, i.e. $\alpha|k| > 1$, $\alpha|p| > 1$ and $\alpha|q| > 1$. In general, we see that the approximation $\mathcal{C}_4$ (strongly) attenuates all interactions for which at least two legs of the triangle $k = p+q$ are (much) longer than $1/\alpha$, whereas all possible triadic interactions for which at least two legs are (much) shorter than $1/\alpha$ are reduced to a small degree. Since in the latter case the longest leg is always shorter than $2/\alpha$, we may conclude that the approximation $\mathcal{C}_4$ confines the dynamics for the greatest part to scales whose wavevector-length is smaller than $2/\alpha$. In this way, the resolution requirements resulting from the convective nonlinearity are reduced.

5 Results for a turbulent channel flow

As a first step in the application of symmetry-preserving regularization, the approximation $\mathcal{C}_4$ is tested for a turbulent channel flow by means of a comparison with the direct numerical simulations performed by Kim et al. [10]. Based on the channel half-width and the friction velocity the Reynolds number is 180. The smooth approximations $\mathcal{C}_n$ given by Eqs. (6)-(8) are constructed such that fundamental properties (3) and (4) are preserved. Of coarse, the same should hold for the numerical approximations that are used to discretize $\mathcal{C}_n$. Therefore, Eq. (5) is discretized as in Ref. [7]. We consider two, coarse, computational grids consisting of $16\times16\times8$ and $32\times32\times16$ grid points, respectively. The filtering is based upon the Helmholtz operator, where the boundary conditions that supplement the Navier-Stokes equations are applied to the filter too. Since solving the Helmholtz equation for $\overline{u}$ is rather expensive, we do not fully solve this equation, but choose to perform just one Jacobi iteration with $\overline{u} = u$ as initial guess.

The least to be expected from a simulation shortcut is a good prediction of the mean flow. As can be seen in Fig. 1, the approximation $\mathcal{C}_4$ satisfies that minimal requirement already at the very coarse $16\times16\times8$ grid, provided the filter length ϵ is taken equal to two-to-four times the grid width h. Yet, the turbulence intensities do not agree so well with the reference data if only 2048 gridpoints are used. Here, it may be noted that the root-mean-square velocity fluctuations are the least worse (in comparison to the DNS of Kim et al. [10]) if the results are extrapolated linearly to $\epsilon = 0$. Overall good agreement between the $\mathcal{C}_4$-calculation at the $32\times32\times16$ grid and the DNS is observed for both the first- and second-order statistics, see Fig. 2. Heuristic arguments as well as computational results (Fig. 3) show that the energy spectrum of the solution of (5)+(7) follows the DNS for large scales of motion, whereas a much steeper (numerically speaking: more gentle) power law is found for small scales.



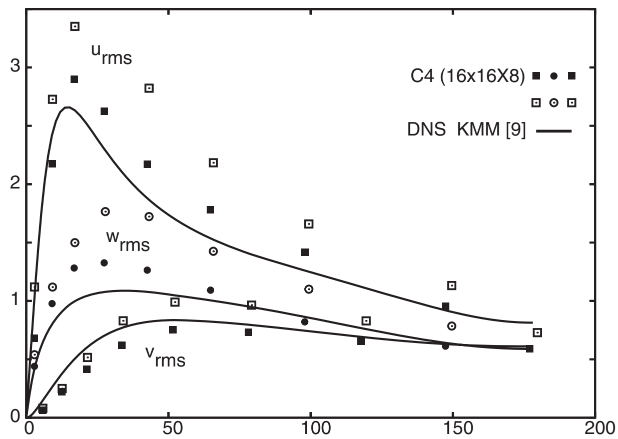
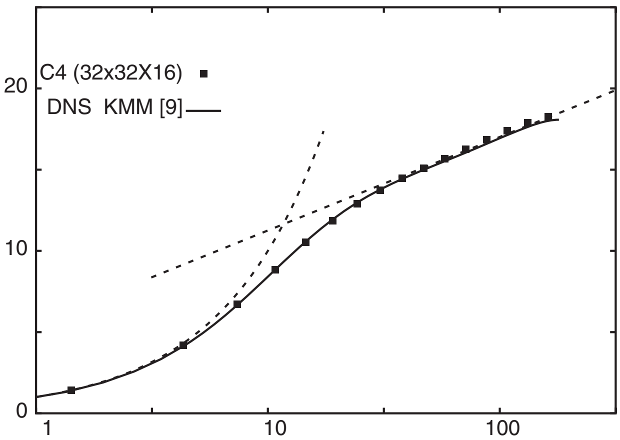


Fig. 1. The upper figure shows the mean velocity (in wall coordinates) for the $16{\times}16{\times}8$ simulations with $\mathcal{C}_4$. The filter length ϵ varies from zero (no regularization) to the four times the grid width h. The lower figure displays the root-mean-square of the fluctuating velocities. The open boxes and circles correspond with $\epsilon = 2.5h$; the filled boxes and circles represent data that is extrapolated linearly to $\epsilon = 0$.

The first results shown here illustrate the potential of symmetry-preserving smoothing as a new simulation shortcut for turbulent channel flow. Yet, given the inherent difficulty of turbulence modeling, more thorough investigations and comparisons need be carried out to clarify the pros and cons.



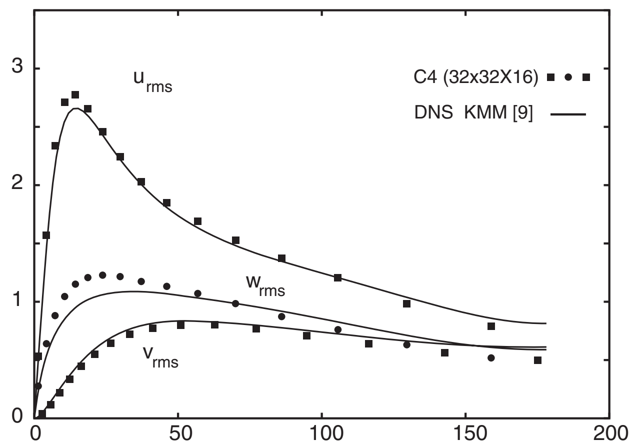


Fig. 2. Results of $\mathcal{C}_4$ with 32×32×16 grid points and $\epsilon = 1.5h$.

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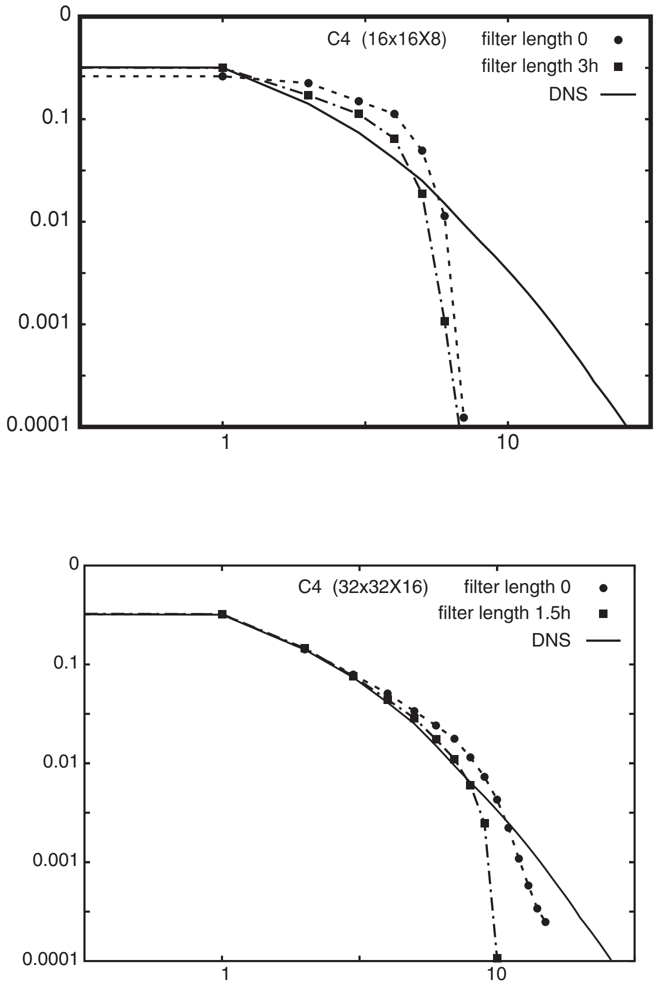

Fig. 3. One-dimensional (streamwise) energy spectra at $y^+ \approx 5$.

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